

Rivlin's Theorem on Walsh Equiconvergence

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1. INTRODUCTION

Recently Rivlin [2] has given a very interesting extension of Walsh's theorem on equiconvergence. Let C denote the complex plane, and let $\mathcal{A}(D(\rho))$, $1 < \rho < \infty$, be the class of functions f that are analytic on the disc $D(\rho) = \{z \in C: |z| < \rho\}$ and have a singularity on the circle $\{z \in C: |z| = \rho\}$. If $f(z) = \sum_0^\infty a_j z^j$, we denote by $S_n(f; z)$ the partial sum $\sum_0^n a_j z^j$. For a positive integer $m = nq + c$, where q, c are fixed integers, let $\omega = e^{2\pi i/m}$. If π_n denotes the family of all polynomials of degree $\leq n$ and if $p_{n,m}(f; z)$ denotes such a polynomial minimizing

$$\sum_{k=0}^{m-1} |f(\omega^k) - q_n(\omega^k)|^2 \quad (1.1)$$

over all polynomials $q_n \in \Pi_n$, then Rivlin proved

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THEOREM A. *Let $f \in \mathcal{A}(D(\rho))$ and let q be a fixed positive integer. Then*

$$\lim_{n \rightarrow \infty} (p_{n,m}(f; z) - S_n(f; z)) = 0, \tag{1.2}$$

for all $z \in D(\rho^{1+q})$, the convergence being uniform and geometric in $|z| \leq \tau < \rho^{1+q}$, where $m = nq + c$, c a fixed integer. Moreover, the result is best possible in the sense that (1.2) fails for every z satisfying $|z| = \rho^{1+q}$ for some $f \in \mathcal{A}(D(\rho))$.

When $m = n + 1$, Theorem A reduces to a well-known theorem of J. L. Walsh [5, 4].

Rivlin gave another extension of Walsh's theorem for functions analytic in the ellipse $\mathcal{E}(\rho)$ in C which is the image of the disc $D(\rho)$ under the mapping $z = \frac{1}{2}(w + w^{-1})$. Let $\mathcal{A}(\mathcal{E}(\rho))$ denote the class of functions f that are analytic on $\mathcal{E}(\rho)$ but not on any region containing the closure of $\mathcal{E}(\rho)$. Let

$$f(z) = \sum_0^{\infty} A_k T_k(z), \tag{1.3}$$

where $T_k(z)$ is the Chebyshev polynomial of degree k and where the prime means that the first term in Eq. (1.3) is to be halved. Let $\xi_j^{(m)}$ ($j = 1, \dots, m$) be the zeros of $T_m(x)$ (i.e., $\xi_j^{(m)} = \cos[(2j - 1)\pi/2m]$, $j = 1, \dots, m$), and let $u_{n,m}(f; z)$ denote the algebraic polynomial which minimizes

$$\sum_{j=1}^m |f(\xi_j^{(m)}) - p_n(\xi_j^{(m)})|^2 \tag{1.4}$$

over all polynomials $p_n \in \Pi_n$. If $S_n(f; z) = \sum_{k=0}^n A_k T_k(z)$, then Rivlin proved

THEOREM B [2]. *If $f \in \mathcal{A}(\mathcal{E}(\rho))$ and q is any integer > 1 , then*

$$\lim_{n \rightarrow \infty} (u_{n,m}(f; z) - S_n(f; z)) = 0, \quad m = nq + c \tag{1.5}$$

for all z in $\mathcal{E}(\rho^{2q-1})$, the convergence being uniform and geometric on $\mathcal{E}(\tau)$ for $\tau < \rho^{2q-1}$.

In addition, Rivlin also showed that Theorem B is also true if we replace $u_{n,m}(f; z)$ by the polynomial $t_{n,m}(f; z)$ which minimizes

$$\sum_{k=1}^m |f(\eta_k^{(m)}) - p_n(\eta_k^{(m)})|^2, \tag{1.6}$$

where $\eta_k^{(m)}$ ($k = 1, \dots, m$) are the extrema of $T_n(x)$ on $[-1, 1]$.

The method of Rivlin is based on the properties of Chebyshev polynomials and their zeros. This makes a further extension of his results difficult. Our purpose here is to propose a mixed problem of interpolation and l_2 -approximation and to extend Rivlin's result in two directions. As a special case we obtain "help" functions which give larger regions of equiconvergence as in [1].

In Section 2 we state the problem and the main results in Theorems 1 and 2. Section 3 deals with the proof of Theorem 1, and the proof of Theorem 2 is given in Section 4.

2. PRELIMINARIES AND MAIN RESULT

Let $A(\rho)$ denote the ring $\{z \in \mathbb{C}: \rho^{-1} < |z| < \rho\}$, $\rho > 1$, and let $\mathcal{A}(A(\rho))$ denote the class of functions f which are analytic on $A(\rho)$ but not on any region containing the closure of $A(\rho)$. Let us set

$$f(z) = \sum_{-\infty}^{\infty} a_j z^j, \quad z \in A(\rho). \tag{2.1}$$

We shall consider the following two problems:

Problem A. For given $f \in \mathcal{A}(A(\rho))$, find the polynomial P_{r+m+n} defined by

$$P_{r+m+n}(z) = P_{r+m+n}(f; z) = \sum_{-r-m-n}^{r+m+n} c_\nu z^\nu \tag{2.2}$$

which satisfies

$$[P_{r+m+n}^{(v)}(\omega^k) - f^{(v)}(\omega^k)] = 0 \quad (v = 0, 1, \dots, r-1, k = 0, 1, \dots, 2m-1), \tag{2.3}$$

where $\omega^{2m} = 1$, and which minimizes

$$\sum_{k=0}^{2m-1} |P_{r+m+n}^{(r)}(\omega^k) - f^{(r)}(\omega^k)|^2, \tag{2.4}$$

over all polynomials of the form (2.2) which satisfy (2.3).

Problem B. Find the region where the difference

$$P_{r+m+n}(f; z) - S_{r+m+n}(f; z) \tag{2.5}$$

tends to zero as $n \rightarrow \infty$, when $m = nq + c$, where c and q are positive integer constants, and where

$$S_{r+m+n}(f; z) = \sum_{-r-m-n}^{r+m+n} a_j z^j \tag{2.6}$$

is a section of the Laurent series (2.1) of f .

The solution to Problem A is given by

THEOREM 1. *The polynomial $P_{rm+n}(f; z)$ of the form (2.2) which satisfies (2.3) and minimizes (2.4) is given by*

$$P_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{C_R} f(t) t^{rm+n} K_1(t, z) dt, \tag{2.7}$$

where

$$\begin{aligned} z^{rm+n} K_1(t, z)(t-z) &= 1 - \left(\frac{z^{2m}-1}{t^{2m}-1} \right)^r \\ &\quad + \frac{(z^{2m}-1)^r}{(t^{2m}-1)^{r+1}} t^{2m-2n-1} (t^{2n+1} - z^{2n+1}), \end{aligned} \tag{2.8}$$

and C_R is the oriented boundary of the ring $A(R)$.

We postpone the proof of Theorem 1 to Section 3 and proceed to state our main result.

THEOREM 2. *If $f \in \mathcal{A}(A(\rho))$, $f(z) = f(1/z)$ for all $z \in A(\rho)$ and $P_{rm+n}(f; z)$ is the solution to Problem A, and if $m = nq + c$, where n, q , and c are positive integers, then*

$$\lim_{n \rightarrow \infty} [P_{rm+n}(f; z) - S_{rm+n}(f; z)] = 0, \tag{2.9}$$

for all $z \in A(\tau(\rho))$, where

$$\begin{aligned} \tau(\rho) &= \rho^{2q-1}, && \text{when } r = 0 \\ &= \min\{\rho^{1+(2q-2)/(qr+1)}, \rho^{1+2/(qr-1)}\}, && \text{when } r \geq 1. \end{aligned} \tag{2.10}$$

Moreover, the convergence is uniform and geometric in any compact subset of the above ring. Also the result is best possible in the sense that (2.9) fails for every z on the boundary of $A(\tau(\rho))$ for some $f \in \mathcal{A}(A(\rho))$.

Remark. Problems A and B can also be formulated and solved in a similar way if instead of considering the minimization problem (2.4) on the zeros of $z^{2m} = 1$, we consider the same problem on the zeros of $z^{2m} = -1$. In this case, ω^k in (2.4) is replaced by $\omega^{k-1/2}$ and the corresponding polynomial $\tilde{P}_{rm+n}(f; z)$, which satisfies (2.3) and (2.4) on the zeros of $z^{2m} = -1$, is given by

$$\frac{1}{2\pi i} \int_{C_R} f(t) t^{rm+n} \tilde{K}_1(t, z) dt.$$

Here $\tilde{K}_1(t, z)$ is obtained from (2.8) by replacing $z^{2m} - 1$ and $t^{2m} - 1$ in (2.8) by $z^{2m} + 1$ and $t^{2m} + 1$, respectively. Also, Theorem 2 holds when $\tilde{P}_{r+m+n}(f; z)$ replaces $P_{r+m+n}(f; z)$.

When $r = 0$, Theorem A gives the polynomials $t_{n,m}(f; z)$ and $u_{n,m}(f; z)$ of Rivlin [2] according as we use the zeros of $z^{2m} + 1$ or of $z^{2m} - 1$ respectively in (2.3) and (2.4).

3. PROOF OF THEOREM 1

Since $P_{r+m+n}(f; z)$ is of the form (2.2) and satisfies (2.3), we have

$$z^{r+m+n} P_{r+m+n}(f; z) = Q_{2rm-1}(z) + (z^{2m} - 1)^r R_{2n}(z), \tag{3.1}$$

where $R_{2n}(z) \in \Pi_{2n}$. From (2.3), we require that

$$[Q_{2rm-1}(z) z^{-rm-n}]_{z=\omega^k}^{(v)} = f^{(v)}(\omega_k) \quad (v = 0, 1, \dots, r-1, k = 0, 1, \dots, 2m-1).$$

Equivalently, we require

$$Q_{2mr-1}^{(v)}(\omega^k) = [z^{r+m+n} f(z)]_{z=\omega^k}^{(v)}.$$

From a known formula [1], we have

$$Q_{2mr-1}(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{r+m+n}}{t-z} \left\{ 1 - \left(\frac{z^{2m}-1}{t^{2m}-1} \right)^r \right\} dt. \tag{3.2}$$

In order to find $P_{r+m+n}^{(r)}(\omega^k)$, we need to evaluate

$$A_1 := [Q_{2rm-1}(z) z^{-rm-n}]_{\omega^k}^{(v)} \quad \text{and} \quad A_2 := [(z^{2m}-1)^r R_{2n}(z) z^{-rm-n}]_{\omega^k}^{(v)}.$$

Since

$$\left[\frac{d^r}{dz^r} (z^{2m}-1)^r \right]_{\omega^k} = r!(2m)^r \omega^{-kr},$$

it is easy to see that

$$A_2 = r!(2m)^r \omega^{-kr} R_{2n}(\omega^k) \omega^{-k(rm+n)} \quad (k = 0, 1, \dots, 2m-1). \tag{3.3}$$

Also from (3.1) and (3.2), we have

$$\begin{aligned} A_1 &= \left[z^{-rm-n} \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{r+m+n}}{t-z} dt \right]_{\omega^k}^{(r)} \\ &\quad - \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{r+m+n}}{(t^{2m}-1)^r} \left[\frac{(z^{2m}-1)^r z^{-rm-n}}{t-z} \right]_{z=\omega^k}^{(r)} dt \\ &= f^{(r)}(\omega^k) - \frac{r!(2m)^r}{2\pi i} \omega^{-k(rm+n+r)} \int_{C_R} \frac{f(t) t^{r+m+n}}{(t^{2m}-1)^r (t-\omega^k)} dt. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), the problem of minimizing (2.4) reduces to minimizing

$$\sum_{k=0}^{2m-1} |R_{2n}(\omega^k) - g(\omega^k)|^2 \tag{3.5}$$

over all polynomials $R_{2n} \in \pi_{2n}$, where

$$g(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{rm+n}}{(t^{2m}-1)^r(t-z)} dt.$$

In order to minimize (3.5), we replace $g(z)$ by its Lagrange interpolant on the $2m$ roots of unity and use a result of Rivlin [2]. Accordingly, the Lagrange interpolant of $g(z)$ is

$$L_{2m-1}(z; g) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{rm+n}(t^{2m}-z^{2m})}{(t^{2m}-1)^{r+1}(t-z)} dt.$$

If $s_{2n}(z; L_{2m-1})$ denotes the Taylor polynomial of degree $2n$ for $L_{2m-1}(z; g)$, the result of Rivlin yields

$$\begin{aligned} R_{2n}(z) &= s_{2n}(z; L_{2m-1}(z; g)) \\ &= \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{mr+2m-n-1}(t^{2n+1}-z^{2n+1})}{(t^{2m}-1)^{r+1}(t-z)} dt. \end{aligned} \tag{3.6}$$

The formula (2.7) is obtained now on using (3.1), (3.2), and (3.6).

COROLLARY 1. *If $f \in \mathcal{A}(A(\rho))$ and if moreover $f(z) = f(z^{-1})$ for all $z \in A(\rho)$, then*

$$\begin{aligned} z^{rm+n} P_{rm+n}(f; z) &= \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t} \left\{ t^{rm+n+1} K_1(t, z) \right. \\ &\quad \left. - \left(\frac{1}{t}\right)^{rm+n+1} K_1\left(\frac{1}{t}, z\right) \right\} dt, \end{aligned} \tag{3.7}$$

where Γ is the circle $|z| = R$, $1 < R < \rho$.

Proof. Since C_R is the union of the circles $|z| = R$ and $|z| = R^{-1}$, a change of variable in the integral on $|z| = R^{-1}$ gives the result after an elementary calculation, because $f(t) = f(t^{-1})$.

Remark. We remark that when $r=0$, $P_n(f; z)$ is the polynomial $t_{n,m}(z; f)$ of Rivlin [2].

Also from (2.6), we see that if $f(t) = f(t^{-1})$, then

$$z^{rm+n} S_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t} K_0(t, z) dt, \tag{3.8}$$

where

$$K_0(t, z) = \left(\frac{z}{t}\right)^{rm+n} \frac{z^{rm+n+1} - t^{rm+n+1}}{z-t} + \frac{1}{t} \frac{z^{rm+n} - (1/t)^{rm+n}}{z - (1/t)}. \tag{3.9}$$

This follows also from the representation of $f(z)$, viz.,

$$f(z) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t)}{t} \left[\frac{t}{t-z} - \frac{t^{-1}}{t^{-1}-z} \right] dt, \tag{3.10}$$

when $f(z) = f(z^{-1})$.

COROLLARY 2. *If $f \in \mathcal{A}(A(\rho))$ and if moreover $f(z) = f(z^{-1})$ for all $z \in A(\rho)$, then*

$$P_{rm+n}(f; z) = P_{rm+n}(f; z^{-1}). \tag{3.11}$$

Proof. From (2.7) and (2.8), we have

$$\begin{aligned} z^{rm+n} P_{rm+n}(f; z) &= \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{rm+n}}{t-z} \\ &\quad \times \left[1 + \frac{(z^{2m}-1)(1-t^{2m-2n-1}z^{2n+1})}{(t^{2m}-1)^{r+1}} \right] dt \end{aligned}$$

and

$$\begin{aligned} \left(\frac{1}{z}\right)^{rm+n} P_{rm+n}(f; z^{-1}) &= \frac{1}{2\pi i} \int_{C_R} \frac{f(t) t^{rm+n}}{t-z^{-1}} \\ &\quad \times \left[1 + \frac{(1-z^{2m})^r z^{-2mr}}{(t^{2m}-1)^{r+1}} \left(1 - \frac{t^{2m-2n-1}}{z^{2n+1}} \right) \right] dt. \end{aligned}$$

Changing t into t^{-1} in the above and simplifying, we have

$$\begin{aligned} P_{rm+n}(f; z^{-1}) &= \frac{1}{2\pi i} \int_{C_R} \left(\frac{z}{t}\right)^{rm+n+1} \\ &\quad \times \left[1 - \frac{(z^{2m}-1)^r z^{-2m(r+1)}}{(t^{2m}-1)^{r+1} z^{2mr}} \left(1 - \frac{z^{-2n-1}}{t^{2m-2n-1}} \right) \right] dt. \end{aligned}$$

From these we obtain after simplifying that

$$P_{rm+n}(f; z) - P_{rm+n}(f; z^{-1}) = \frac{1}{2\pi i} \int_{C_R} \frac{f(t)}{t-z} \left[\left(\frac{t}{z}\right)^{rm+n} - \left(\frac{z}{t}\right)^{rm+n} \right] dt = 0,$$

because the integrand is single-valued analytic in the annulus C_R .

4. SOME LEMMAS AND PROOF OF THEOREM 2

The proof of Theorem 2 will require a number of estimates and to this effect we prove

LEMMA 1. *If $f(z)$ satisfies the conditions of Theorem 2, then we have*

$$P_{rm+n}(f; z) - S_{rm+n}(f; z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t} A(t, z) dt, \tag{4.1}$$

where $A(t, z)$ is given by

$$A(t, z) = S_1(t, z) + S_2(t, z) + S_3(t, z) - S_2(t^{-1}, z) - S_3(t^{-1}, z) \tag{4.2}$$

and

$$\begin{aligned} S_1(t, z) &= -\frac{(t^{2mr+2n+1} - z^{2mr+2n+1})}{(t-z)(tz)^{rm+n}}, \\ S_2(t, z) &= \frac{t^{mr+n+1} \{ (t^{2m}-1)^r - (z^{2m}-1)^r \}}{(t^{2m}-1)^r (t-z) z^{rm+n}}, \\ S_3(t, z) &= \frac{(z^{2m}-1)^r t^{mr+2m-n} (t^{2n+1} - z^{2n+1})}{(t^{2m}-1)^{r+1} (t-z) z^{rm+n}}. \end{aligned} \tag{4.3}$$

Proof. These formulae are obtained from (3.7) and (3.8) and on adding the integral

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f(t)}{t} \left\{ \frac{z^{rm+n} - t^{rm+n}}{t-z} + \frac{1}{t} \frac{z^{rm+n} - (1/t)^{rm+n}}{z-t^{-1}} \right\} dt$$

to the right side of (3.7), since it is easily seen to be zero when $f(t) = f(1/t)$.

LEMMA 2. *The following identity holds:*

$$\frac{(t^{2m}-1)^r - (z^{2m}-1)^r}{t-z} = \sum_{k=0}^{2mr-1} z^k t^{-k-1} A_k(t), \tag{4.4}$$

where $A_k(t)$ is a polynomial such that

$$\begin{aligned} A_{2mv}(t) &= A_{2mv+1}(t) = \dots = A_{2m(v+1)-1}(t) \\ &= (t^{2m}-1)^r - \sum_{j=0}^v (-1)^{r-j} \binom{r}{j} t^{2mj} \\ &= \sum_{j=v+1}^r (-1)^{r-j} \binom{r}{j} t^{2mj} \quad (v=0, 1, \dots, r-1). \end{aligned} \tag{4.5}$$

This is easily verified. When $r = 0$, $A_k(t)$'s are all zero, and when $r = 1$, $A_k(t) = t^{2m}$.

LEMMA 3. *If we set*

$$A(t, z) = \sum_{j = -mr - n}^{mr + n} \lambda_j(t) z^j, \tag{4.6}$$

then $\lambda_j(t) = \lambda_{-j}(t)$, $j = 1, 2, \dots, rm + n$, and for $|t| = R$ ($1 < R < \rho$), we have

$$\begin{aligned} \lambda_{|j|}(t) &= O(R^{-mr - n - 1}), & m(r - 2\lambda - 2) + n + 1 \leq |j| \leq m(r - 2\lambda) - n - 1 \\ &= O(R^{-mr - 2m + n}), & \max(0, m(r - 2\lambda) - n) \leq |j| \leq m(r - 2\lambda) + n. \end{aligned} \tag{4.7}$$

The proof of this lemma depends on Lemma 2 and (4.3). The estimates (4.7) can be used to prove Theorem 2, but we provide here a simple proof.

Proof of Theorem 2. Set $A := S_1(t, z) + S_2(t, z) + S_3(t, z)$. Then from (4.3) we obtain

$$\begin{aligned} A &= \frac{z^{mr + n + 1}}{(t - z) t^{mr + n}} - \frac{(1 - z^{-2m})^r}{(1 - t^{-2m})^{r+1}} \frac{z^{mr - n}}{t^{mr + 2m - n - 1}} \frac{(-1 + z^{2n + 1} t^{2m - 2n - 1})}{(t - z)} \\ &= \frac{1}{t - z} \left[\frac{z^{mr + n + 1}}{t^{mr + n}} + \frac{z^{mr - n} (1 - z^{-2m})^r}{t^{mr + 2m - n - 1} (1 - t^{-2m})^{r+1}} \right. \\ &\quad \left. - \frac{z^{mr + n + 1}}{t^{mr + n}} \frac{(1 - z^{-2m})^r}{(1 - t^{-2m})^{r+1}} \right] \\ &= \frac{1}{t - z} \left[\frac{z^{mr + n + 1}}{t^{mr + n}} \{O(z^{-2m}) + O(t^{-2m})\} \right. \\ &\quad \left. + \frac{z^{mr - n}}{t^{mr + 2m - n - 1}} \{1 + O(z^{-2m}) + O(t^{-2m})\} \right], \end{aligned}$$

where we may assume without loss of generality that $|z| > 1$, $|t| > 1$. Moreover, we observe that if we set $B := S_1(1/t, z) + S_3(1/t, z)$, then using (4.3) again we see that

$$\begin{aligned} B &= \frac{t^{-mr - n - 1} \{(t^{-2m} - 1)^r - (z^{2m} - 1)^r\}}{(t^{-2m} - 1)^r (t^{-1} - z) z^{mr + n}} \\ &\quad + \frac{(z^{2m} - 1)^r}{(t^{-2m} - 1)^{r+1}} \frac{t^{-mr - 2m + n} (t^{-2n - 1} - z^{2n + 1})}{(t^{-1} - z) z^{mr + n}}. \end{aligned}$$

Some simplification yields

$$B = \frac{1}{1-tz} \left[\frac{1}{(tz)^{mr+n}} - \frac{(-1)^r z^{mr-n}}{t^{mr+n}} (1 + O(z^{-2m}) + O(t^{-2m})) \right. \\ \left. + \frac{(-1)^r z^{mr+n+1}}{t^{mr+2m-n-1}} (1 + O(z^{-2m}) + O(t^{-2m})) \right].$$

From the above estimates for A and B we see that as $n \rightarrow \infty$,

$$A = O\left(\frac{z^{mr+n}}{t^{mr+n+2m}}\right) + O\left(\frac{z^{mr-n}}{t^{mr+2m-n-1}}\right)$$

and

$$B = O\left(\frac{z^{mr-n}}{t^{mr+n}}\right) + O\left(\frac{z^{mr+n+1}}{t^{mr+2m-n-1}}\right).$$

Thus we have

$$A(t, z) = O\left(\frac{z^{mr-n}}{t^{mr+n}}\right) + O\left(\frac{z^{mr+n+1}}{t^{mr+2m-n-1}}\right), \tag{4.8}$$

which tends to zero if

$$|z| < \min\{\rho^{(mr+n)/(mr-n)}, \rho^{(mr+2m-n-1)/(mr+n+1)}\}.$$

This gives the result when $n \rightarrow \infty$ and completes the proof for $r > 1$.

For $r=0$, $S_2(t, z)$ and $S_2(1/t, z)$ do not occur and the estimate in Theorem 2 is easily obtained from (4.8), since in this case $A(t, z) = O(z^{n+1}/t^{2m-n-1})$.

Remark. For $r=0$, the polynomial $P_n(z)$ in Theorem 1 can be easily seen to be the polynomial $t_{n,m(n)}(z, f)$ of Rivlin [2]. In fact we can see from (2.7) and (2.8) that

$$P_n(z) = t_{n,m(n)}(z, f) = \sum_{j=0}^n T_j(z) \frac{1}{2\pi i} \int_{\Gamma} f(t) \left(\frac{t^{2m-j-1}}{t^{2m-1}} + \frac{t^{j-1}}{t^{2m-1}} \right) dt.$$

If we set

$$s_{n,v}(z, f) = \sum_{j=0}^n (A_{2vm+j} + A_{2vm-j}) T_j(z) \quad (v = 1, 2, 3, \dots),$$

where $f(z)$ is given by (1.3), then

$$\lim_{n \rightarrow \infty} t_{n,m}(z; f) - s_n(z; f) - \sum_{v=1}^{l-1} s_{n,v}(z; f) = 0 \tag{4.9}$$

for $|z| < \rho^{2lq-1}$.

Theorems 1 and 2 can be formulated for functions in $\mathcal{A}(A(\rho))$ and an analogue of (4.9) can also be obtained from the representation (4.1).

It would be interesting to obtain sharpness results analogous to those of Saff and Varga [3] and the analogue of Theorem 2 above when Hermite interpolation is replaced by lacunary interpolation as in [6].

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